GENERATION OF TWO-DIMENSIONAL WATER WAVES BY MOVING BOTTOM DISTURBANCES

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ABSTRACT. In this study we investigate the potential and limitations of the wave generation by disturbances moving at the bottom. More precisely, we assume that the wavemaker is composed of an underwater object of a given shape which can be displaced according to a given trajectory. The practical question we address in this study is how to compute the wavemaker shape and its trajectory in order to generate a wave with prescribed characteristics? For the sake of simplicity we model the hydrodynamics by a generalized forced BBM equation. This practical problem is reformulated as a constrained nonlinear optimization problem. Some constraints are imposed in order to make practically feasible the computed solution. Finally, we show some numerical results to support our theoretical and algorithmic developments.

CONTENTS

1. INTRODUCTION

The problem of wave generation is complex and has many practical applications. On the scale of a laboratory wave tank a traditional wavemaker is composed of numerous paddles attached to a vertical wall and which can move independently according to some prescribed program. These systems have been successfully used to conduct laboratory experiments at least since late 60's [25, 37].

In this study we investigate theoretically and numerically the potential for practical applications of a different kind of wave making devices. Namely, the mechanism considered in this study is composed mainly of an underwater object which can be displaced along a portion of the bottom with the prescribed trajectory. In mathematical terms, we study

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FIGURE 1. An artificial wave generated in a pool by an underwater wave-making device. © http://www.wavegarden.com/

the wave excitation problem by moving forcing. Similar processes are known in physics under the name of autoresonance phenomena, thoroughly studied by L. FRIEDLAND and his collaborators [26, 27, 28].

Recently, this type of wavemakers have found an interesting application to the man-made surfing facilities [34]. This type of devices was proven to be successful to generate high quality waves for surfing far from the Oceans. Our main goal consists in providing the first elements of theoretical explanation of this process. The second objective of this study is to provide an efficient computational procedure to determine the underwater object shape and trajectory to generate a prescribed wave profile in a given portion of the wave tank.

The problem of wave generation by moving bottom has been particularly studied in the context of tsunami wave generation. These extreme waves are caused by sea bed displacements due to an underwater earthquake [32, 9, 41, 19, 24] or a submarine landslide [60, 42, 61, 2]. However, the bottom motion produced by an underwater earthquake is mainly vertical, even if some effort has been made to include also the horizontal displacements components [55, 57, 56, 40, 23]. On the other hand, the wave making mechanism

studied here involves only the horizontal motion. Consequently, the knowledge of the tsunami wave community cannot be directly transposed to this problem.

The wave propagation takes place in a shallow channel, so the long wave assumption can be adopted [59, 15]. However, weak dispersive and weak nonlinear effects should be included since the resulting wave observed in experiments has some common characteristics with a solitary wave. Consequently, as the base model we choose the classical Boussinesq system derived by D.H. Peregrine (1967) [43] and generalized later by T. Wu (1987) [62], who included the time-dependent bathymetry effects. In order to simplify further the problem, we assume the wave propagation to be unidirectional and, hence, we derive a generalized forced BBM equation [3]. This equation is then discretized with a high order finite volume method [4, 21, 12, 22]. Finally, the trajectory and the shape of the underwater object are optimized in order to minimize a cost-function under some practical feasibility constraints.

From mathematical theoretical point of view our formulation can be seen as the controllability problem for the forced BBM equation [45]. Let us describe the main available results on the controllability of dispersive wave equations such as KdV [8, 36], BBM [3] and some Boussinesq-type systems [6].

The controllability of the KdV equation:

$$u_t + u_{xxx} + u_x + uu_x = 0, \qquad x \in [0, L], t > 0$$

is well studied in the literature. The controllability and stabilization properties were obtained by L. Russell & B.-Y. Zhang (1993) [50] for periodic boundary conditions with an internal control. The boundary control was investigated by the same authors later [51]. The controllability of the KdV equation with Dirichlet boundary conditions was studied in the following papers [46, 63, 44, 13, 48, 10, 30, 11, 31]. This list of references is not exhaustive.

Let us briefly describe now some results on the controllability of the BBM equation. Rosier & Zhang (2012) [49] proved the Unique Continuation Property (UCP) for the BBM equation posed on a one dimensional torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$:

$$u_t - u_{txx} + u_x + uu_x = 0, \tag{1.1}$$

i.e. for any open nonempty set $\omega \subset \mathbb{T}$ the only solution of (1.1) with

$$u(x,t) = 0 \text{ for } (x,t) \in \omega \times (0,T)$$

is the trivial solution $u = 0$. Moreover, they proved the UCP also for BBM-type equations of the form

$$u_t - u_{txx} + [f(u)]_x = 0,$$

where $f \in C^1(\mathbb{R}), f(u) \geq 0$ for all $u \in \mathbb{R}$, and the only solution $u \in (-\delta, \delta)$ of $f(u) = 0$ is $u = 0$, for some number $\delta > 0$. Furthermore, they consider the following control problem:

$$u_t - u_{txx} + u_x + uu_x = a(x+ct)h(x,t),$$

where $a \in C^\infty$ is given and $h(x,t)$ is the control. They prove local exact controllability in $H^s(\mathbb{T})$ for any $s \geq 0$ and global exact controllability in $H^s(\mathbb{T})$ for any $s \geq 1$. A necessary

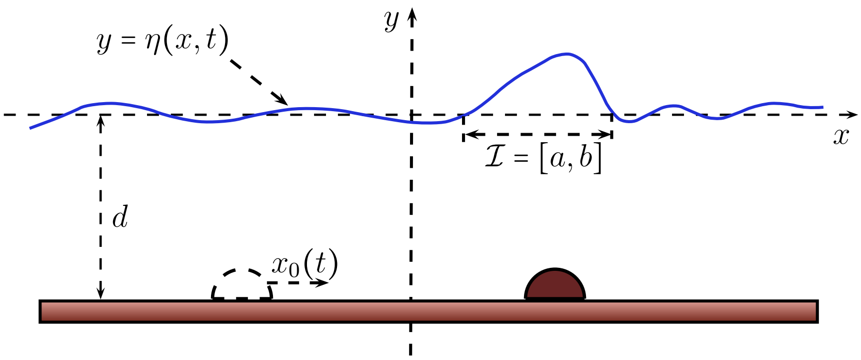


FIGURE 2. Sketch of the physical domain with an underwater object moving along the bottom.

and sufficient algebraic condition for approximate controllability of the BBM equation with homogeneous Dirichlet boundary conditions was given in [1]. The controllability of linearized BBM and KdV equations was studied in [47, 38, 65]. The controllability of a family of Boussinesq equations have been studied theoretically as well [64]. In [58], TOUBOUL recently obtained controllability results for the heat and wave equations with a moving boundary.

The present study is organized as follows. In Section 2 we derive the governing equation. Then, this model is analysed mathematically in Section 3. The results of some numerical simulations are presented in Section 4. Finally, in Section 5 we outline the main conclusions of this study.

2. MATHEMATICAL MODEL

Consider an ideal incompressible fluid of constant density in a two-dimensional domain. The horizontal independent variable is denoted by x and the upward vertical one by y. The origin of the Cartesian coordinate system is chosen such that the line $y = 0$ corresponds to the still water level. The fluid is bounded below by an impermeable bottom at $y = -h(x,t)$ and above by an impermeable free surface at $y = \eta(x,t)$. We assume that the total depth $H(x,t) \equiv h(x,t) + \eta(x,t)$ remains positive $H(x,t) \geqslant H_0 > 0$ at all times t. The sketch of the physical domain is shown in Figure 2. The depth-averaged horizontal velocity is denoted by $u(x,t)$ and the gravity acceleration by g. The fluid layer has the uniform depth d everywhere, which is perturbed only by a localized object, which can move along the bottom:

$$h(x,t) = d - \zeta(x,t), \quad \zeta(x,t) = \zeta_0(x - x_0(t)), \tag{2.1}$$

where the function $\zeta_0(x)$ has a compact support and $x = x_0(t)$ is the trajectory of its barycenter. The meaning of the segment $\mathcal{I} = [a,b]$ is explained in Section 3.1.

In 1987 T. Wu [62] derived the following Boussinesq-type system to study the generation of solitary waves by moving disturbances:

$$\eta_t + ((h+\eta)\,u)_x = -h_t, \tag{2.2}$$

$$u_t + uu_x + g\eta_x = \frac{1}{2}h\,(h_t + (hu)_x)_{xt} - \frac{1}{6}h^2 u_{xxt}. \tag{2.3}$$

This system represents a further generalization of the classical Boussinesq equations derived by D.H. Peregrine (1967) [43] for the case of the moving bottom $h(x,t)$. In our work we take the system (2.2), (2.3) as the starting point. In order to simplify it further, we will switch to dimensionless variables (denoted with primes):

$$x' = \frac{x}{l}, \quad \eta' = \frac{\eta}{a}, \quad h' = \frac{h}{d}, \quad u' = \frac{u}{\frac{ga}{\sqrt{gd}}}, \quad t' = \frac{t}{\frac{\ell}{\sqrt{gd}}}, \text{ and } \zeta' = \frac{\zeta}{a},$$

where a and ℓ are the characteristic wave amplitude and wavelength correspondingly. We can compose three important dimensionless numbers which characterize the Boussinesq regime:

$$\varepsilon := \frac{a}{d} \ll 1, \quad \mu^2 := \left(\frac{d}{\ell}\right)^2 \ll 1, \text{ and } S := \frac{\varepsilon}{\mu^2} = O(1),$$

where S is the so-called Stokes-Ursell number [59], which measures the relative importance of dispersive and nonlinear effects. In unscaled variables the Peregrine-Wu system takes the following form (for simplicity we drop out the primes):

$$\eta_t + ((h+\varepsilon\eta)\,u)_x = -h_t, \tag{2.4}$$

$$u_t + \varepsilon uu_x + \eta_x = \frac{\mu^2}{2}h\,(h_t + (hu)_x)_{xt} - \frac{\mu^2}{6}h^2 u_{xxt}. \tag{2.5}$$

To simplify the problem, we will reduce the Boussinesq system (2.2), (2.3) to the unidirectional wave propagation. For instance, in the original work of T. Wu [62] a similar reduction to a forced KdV is also performed. However, the resulting model in our work will be of the BBM-type [3], since it possesses better numerical stability properties.

The reduction to the BBM equation can be done in the following way [35]. The horizontal velocity u can be approximatively represented in unscaled variables as

$$u = +\eta + \varepsilon P + \mu^2 Q + O(\varepsilon^2 + \varepsilon\mu^2 + \mu^4), \tag{2.6}$$

where $P(x,t)$ and $Q(x,t)$ are some functions to be determined. The sign + in front of η means that we consider the waves moving in the rightward direction. Substituting the representation (2.6) into unscaled Boussinesq equations (2.4), (2.5) yields two equivalent relations:

$$\eta_t + \eta_x + \varepsilon P_x + \mu^2 Q_x + 2\varepsilon\eta\eta_x - \varepsilon(\zeta\eta)_x = O(\varepsilon^2 + \varepsilon\mu^2 + \mu^4), \tag{2.7}$$

$$\eta_t + \eta_x + \varepsilon P_t + \mu^2 Q_t + \varepsilon\eta\eta_x = \frac{\mu^2}{2}h\,(-\zeta_t + (h\eta)_x)_{xt} - \frac{\mu^2}{6}h^2\eta_{xxt} + O(\varepsilon^2 + \varepsilon\mu^2 + \mu^4). \tag{2.8}$$

By subtracting two last asymptotic relations we obtain the following compatibility condition:

$$\varepsilon\left(P_x - P_t\right) + \mu^2\left(Q_x - Q_t\right) + \varepsilon\eta\eta_x - \varepsilon\left(\zeta\eta\right)_x = -\frac{\mu^2}{2}\left(-\zeta_t + \eta_x\right)_{xt} + \frac{\mu^2}{6}h^2\eta_{xxt} + O(\varepsilon^2 + \varepsilon\mu^2 + \mu^4). \quad (2.9)$$

For the right-going waves we have the following identities:

$$P_t = -P_x + O(\varepsilon), \qquad Q_t = -Q_x + O(\varepsilon).$$

Finally, the unknown functions P_x and Q_x can be chosen to satisfy asymptotically the compatibility condition (2.9), which yields

$$2P_x = (\zeta\eta)_x - \eta\eta_x, \qquad 2Q_x = \frac{1}{2}\zeta_{xtt} - \frac{1}{3}\eta_{xxt},$$

where we used also the analytical representation of $h(x,t) = 1 - \varepsilon\zeta(x,t)$. The BBM equation in unscaled variables can be now easily obtained by substituting expressions for P_x and Q_x into equation (2.7):

$$\eta_t + \eta_x + \frac{\varepsilon}{2}\left((\zeta\eta)_x - \eta\eta_x\right) + \frac{\mu^2}{2}\left(\frac{1}{2}\zeta_{xtt} - \frac{1}{3}\eta_{xxt}\right) + 2\varepsilon\eta\eta_x - \varepsilon(\zeta\eta)_x = 0.$$

Turning back to physical variables, the generalized forced BBM (gBBM) equation takes the following form:

$$\eta_t + \left(\sqrt{gd}\,\eta + \sqrt{\frac{g}{d}}\left(\frac{3}{4}\eta^2 - \frac{1}{2}\zeta\eta\right)\right)_x - \frac{d^2}{6}\eta_{xxt} = -\frac{1}{4}\frac{d^2}{\sqrt{gd}}\zeta_{xtt}.$$

In subsequent sections we will use this equation to model wave-bottom interaction. However, in order to simplify the notations, we will introduce a new set of dimensionless variables, where all the lengthes are unscaled with the water depth d, velocities with $\sqrt{gd}$ and the time variable with $\sqrt{\dfrac{d}{g}}$. In this scaling the gBBM equation reads:

$$\eta_t + \left(\eta + \frac{3}{4}\eta^2 - \frac{1}{2}\zeta\eta\right)_x - \frac{1}{6}\eta_{xxt} = -\frac{1}{4}\zeta_{xtt}. \quad (2.10)$$

Recall that here η is the unknown free surface elevation, and ζ is a given function, which is the topography of moving body defined by (2.1). The last equation has to be completed by appropriate initial and boundary conditions (when posed on a finite or semi-infinite domain):

$$\eta(x,0) = \eta_0(x), \quad x \in \mathbb{R}. \quad (2.11)$$

3. Well-posedness of the gBBM equation

In this section we give a proof of the well-posedness of gBBM equation (2.10) in the Sobolev spaces $H^s := H^s(\mathbb{R})$. First, we have the following result of the well-posedness of this problem.

Theorem 1. *For any $\zeta \in C^2([0,\infty), H^s) \cap C([0,\infty), H^{[s]+1})$ and $\eta_0 \in H^s$, $s \geq 0$ the problem (2.10), (2.11) admits a unique solution $\eta \in C([0,\infty), H^s)$.*

Proof. **Uniqueness.** Let us assume that for some given functions ζ and η_0 our problem (2.10), (2.11) admits two different solutions η_1 and η_2. The difference $\tilde{\eta} := \eta_1 - \eta_2$ satisfies the following initial-value problem:

$$\tilde{\eta}_t + \left(\tilde{\eta} + \frac{3}{4}\tilde{\eta}(\eta_1 + \eta_2) - \frac{1}{2}\zeta\tilde{\eta}\right)_x - \frac{1}{6}\tilde{\eta}_{xxt} = 0, \qquad \tilde{\eta}(x,0) = 0. \tag{3.1}$$

Taking the scalar product of (3.1) with η in L^2, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left(\frac{1}{2}\tilde{\eta}^2(x,t) + \frac{1}{12}\tilde{\eta}_x^2(x,t)\right)\mathrm{d}x = \int_{\mathbb{R}} \left(\frac{3}{4}\tilde{\eta}(\eta_1+\eta_2) - \frac{1}{2}\zeta\tilde{\eta}\right)\tilde{\eta}_x \mathrm{d}x.$$

The Sobolev and Hölder inequalities imply

$$\left|\int_{\mathbb{R}} \left(\frac{3}{4}\tilde{\eta}(\eta_1+\eta_2) - \frac{1}{2}\zeta\tilde{\eta}\right)\tilde{\eta}_x \mathrm{d}x\right| \le C\left(\|\eta_1\|_0 + \|\eta_2\|_0 + \|\zeta\|_0\right)\|\tilde{\eta}\|_1^2.$$

Thus, the application of the Gronwall inequality yields $\tilde{\eta} = 0$. □

Existence. For any fixed time horizon $T > 0$ let us show that our problem (2.10), (2.11) has a solution $\eta \in C([0,T), H^s)$. J. Bona & N. Tzvetkov (2009) [7] proved that for any given initial data $\eta_0 \in H^s$ the following BBM equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \qquad u(x,0) = u_0(x),$$

admits a unique solution $u \in C([0,\infty), H^s)$ (for $s < 0$ they proved that the system is ill-posed). Using a scaling argument, we have also the well-posedness of the same equation with some positive coefficients:

$$u_t + \left(u + \frac{3}{4}u^2\right)_x - \frac{1}{6}u_{xxt} = 0, \qquad u(x,0) = \eta_0(x). \tag{3.2}$$

We seek a solution of (2.10), (2.11) in the form $\eta = u + v$, where $u \in C([0,\infty), H^s)$ is the solution of (3.2) and v satisfies

$$v_t + \left(v + \frac{3}{4}(v^2 + 2uv) - \frac{1}{2}\zeta(v+u)\right)_x - \frac{1}{6}v_{xxt} = -\frac{1}{4}\zeta_{xtt}, \qquad v(x,0) = 0.$$

Let us prove the existence of such $v \in C([0,\infty), H^s)$ by induction on $[s]$. First, we assume $[s] = 0$. Taking the scalar product of (3) with v in L^2, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left(\frac{1}{2}v^2(x,t) + \frac{1}{12}v_x^2(x,t)\right)\mathrm{d}x = -\int_{\mathbb{R}} \frac{1}{4}\zeta_{xtt} v \,\mathrm{d}x + \int_{\mathbb{R}} \left(\frac{3}{2}uv - \frac{1}{2}\zeta(v+u)\right)v_x \mathrm{d}x.$$

Using the Sobolev and Hölder inequalities, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(\cdot,t)\|_1^2 \le C\Big(\|\zeta_{tt}(\cdot,t)\|_0\|v(\cdot,t)\|_1 + \|u(\cdot,t)\|_0\|v(\cdot,t)\|_1^2 + \|\zeta(\cdot,t)\|_1\|v(\cdot,t)\|_1\left(\|v(\cdot,t)\|_0 + \|u(\cdot,t)\|_0\right)\Big). \tag{3.3}$$

After integrating (3.3) on the interval $(0,t)$ we obtain

$$\|v(\cdot,t)\|_1^2 \le C \sup_{t\in[0,T]} \|v(\cdot,t)\|_1 \int_0^T \Big(\|\zeta_{tt}(\cdot,s)\|_0 + \|u(\cdot,s)\|_0 \|v(\cdot,s)\|_1 + \\ \|\zeta(\cdot,s)\|_1 \left(\|v(\cdot,s)\|_0 + \|u(\cdot,s)\|_0\right) \Big) \mathrm{d}s, \quad (3.4)$$

which is valid for any $t\in[0,T]$. Hence, $\sup_{t\in[0,T]} \|v(\cdot,t)\|_1^2$ also can be estimated by the right hand-side of (3.4). Dividing by $\sup_{t\in[0,T]} \|v(\cdot,t)\|_1^2$ and applying the Gronwall inequality, we deduce

$$\sup_{t\in[0,T]} \|v(\cdot,t)\|_1 \le C \int_0^T \left(\|\zeta_{tt}(\cdot,s)\|_0 + \|\zeta(\cdot,s)\|_1 \|u(\cdot,s)\|_0\right) \mathrm{d}s \times \\ \exp\left(\int_0^T (\|u(\cdot,s)\|_0 + \|\zeta(\cdot,s)\|_1) \mathrm{d}s \right).$$

Using this estimation and applying Picard iteration scheme, we obtain existence of the solution in the case $s\in[0,1]$.

By induction, now we assume the existence of v for $[s]<\alpha$ for some integer $\alpha>1$ and let us prove it for $[s]=\alpha$. To this end, let us take the $\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}$, $\alpha\le[s]$ derivative of (3), multiply the resulting equation by $v_\alpha := \frac{\mathrm{d}^\alpha v}{\mathrm{d}x^\alpha}$ and integrating in x over $\mathbb{R}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \left(\frac{1}{2} v_\alpha^2(x,t) + \frac{1}{12} v_{\alpha x}^2(x,t) \right) \mathrm{d}x = -\int_{\mathbb{R}} \left(\frac{1}{4} \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} \zeta_{xtt} \right) v_\alpha \right) \mathrm{d}x + \\ \int_{\mathbb{R}} \left(\frac{3}{4} \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} (v^2 + 2uv) - \frac{1}{2} \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} (\zeta(v+u)) \right) v_{\alpha x} \mathrm{d}x. \quad (3.5)$$

All the terms can be treated as above, except the term $\int_{\mathbb{R}} \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}(v^2) v_{\alpha x} \mathrm{d}x$, which is not zero in general. Using the induction hypothesis and the fact that $\alpha-1\ge1$, we can estimate

$$\left| \int_{\mathbb{R}} \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}(v^2) v_{\alpha x} \mathrm{d}x \right| \le C \|v_\alpha\|_1^2 \|v\|_{\alpha-1} < M \|v_\alpha\|_1^2.$$

Using the last estimation along with (3.5), the Sobolev and Hölder inequalities, as above, we prove the required estimation for v_α, which completes the proof. □

3.1. **Optimization problem.** In this section we turn to the optimization problem for the gBBM equation (2.10). We assume that the wavemaking piston is a solid, non-deformable object. Thus, its shape, given by a localized function $\zeta_0(x)$, is preserved during the motion and it is sufficient to prescribe the trajectory of its barycenter only $x = x_0(t)$. Consequently, the time-dependent bathymetry is given by the following equation

$$h(x,t) = d - \zeta_0\big(x - x_0(t)\big).$$

The piston shape $\zeta_0(x)$ and its trajectory $x_0(t)$ will be determined as a solution of the optimization problem. More precisely, in the next section we will find numerically these functions in order to produce the largest possible wave (in L_2 sense) in a given subinterval $\mathcal{I} = [a,b]$ of the numerical wave tank at some fixed time $T > 0$. In other words, we minimize the following functional:

$$J(x_0,\zeta_0) = -\int_{\mathcal{I}} \eta(x,T)^2 \mathrm{d}x \longrightarrow \min, \tag{3.6}$$

where $\eta(x,t)$ is the solution of (2.10), (2.11). The existence of this solution is proven in the following

Theorem 2. *For any constants $\varepsilon, M > 0$, there exists $(x_0^*,\zeta_0^*) \in B_M$ such that*

$$J(x_0^*,\zeta_0^*) = \inf_{(x_0,\zeta_0)\in B_M} J(x_0,\zeta_0),$$

where B_M is a closed ball in $H^{2+\varepsilon}[0,T] \times H_0^{2+\varepsilon}([0,1])$ centered at origin with radius M .

Proof. Let (x_0^n,ζ_0^n) be an arbitrary minimizing sequence of J. Since $H^{2+\varepsilon}[0,T]\times H_0^{2+\varepsilon}([0,1])$ is Hilbert space, extracting a subsequence, if it is necessary, we can assume that there is $(x_0^*,\zeta_0^*) \in B_M$ such that $(x_0^n,\zeta_0^n) \rightharpoonup (x_0^*,\zeta_0^*)$ in B_M.

Let us denote η^n the solution of (2.10), (2.11) with $\zeta = \zeta^n := \zeta_0^n(x - x_0^n(t))$. Let us show that we have $\eta^n(T) \to \eta^*(T)$ in L^2, where η^* is the solution of (2.10), (2.11) with $\zeta = \zeta^*$. Indeed, for $\tilde{\eta}^{n,m} := \eta^n - \eta^m$ we have

$$\tilde{\eta}_t^{n,m} + \left(\tilde{\eta}^{n,m} + \frac{3}{4}\tilde{\eta}^{n,m}(\eta^n + \eta^m) - \frac{1}{2}\zeta^n\tilde{\eta}^{n,m} - \frac{1}{2}\zeta^{n,m}\tilde{\eta}^m\right)_x - \frac{1}{6}\tilde{\eta}_{xxt}^{n,m} = -\frac{1}{4}\tilde{\zeta}_{xtt}^{n,m}. \qquad \tilde{\eta}(x,0) = 0. \tag{3.7}$$

Since $\zeta_{tt}^n \to \zeta_{tt}^* := \partial_{tt}(\zeta_0^*(x - x_0^*(t)))$ in $L^2([0,T],L^2)$, multiplying (3.7) in L^2 by $\tilde{\eta}^{n,m}$, integrating by parts and applying the Gronwall inequality, we obtain that η^n is a Cauchy sequence in H^1. Hence,

$$J(x_0^*,\zeta_0^*) = \lim_{n\to\infty} J(x_0^n,\zeta_0^n).$$

This completes the proof of the theorem. □

However, in practice, the functional (3.6) has to be completed by appropriate constraints in order to provide a solution realizable in practice. For example, the speed of the underwater piston is limited by technological and energy consumption constraints. Some more realistic formulations will be addressed numerically in the next Section.

4. Numerical results

In order to discretize the gBBM equation (2.10), posed on a finite interval $[\alpha,\beta]$, we use a modern high-order finite volume scheme with the FVCF flux [29] and the UNO2 reconstruction [33]. The combination of these numerical ingredients has been extensively tested and validated in the context of the unidirectional wave models [22] and Boussinesq-type equations [20, 21]. For the time-discretization, we use the third-order Runge-Kutta scheme,

Parameter	*Value*
Computational domain $[\alpha, \beta]$:	$[-5, 10]\,\mathsf{m}$
Wave quality evaluation area $[a, b]$:	$[0, 6]\,\mathsf{m}$
Number of discretization points N:	1000
CFL number:	1.95
Gravity acceleration g:	$9.8\,\mathsf{m\,s^{-2}}$
Undisturbed water depth d:	$1.0\,\mathsf{m}$
Final simulation time T:	$7.0\,\mathsf{s}$
Piston motion total time T_f:	$3.0\,\mathsf{s}$
Piston length ℓ_0:	$1.0\,\mathsf{m}$
Piston maximal height a_0/d:	0.1
Piston starting point x_0^0:	$x = 0.0\,\mathsf{m}$
Upper bound of the piston position x_{max}:	$x_{max} = 4.5\,\mathsf{m}$
Upper bound of the piston speed v_f:	$1.5\,\mathsf{m/s}$
Wave generation limit x_f:	$x = 1.0\,\mathsf{m}$
$\mathcal{N}$-wave solution ceter x_m:	$x = 2.0\,\mathsf{m}$

TABLE 1. Values of various parameters used in numerical computations.

which is also used in the `ode23` function in Matlab [52]. In all experiments presented below we assume that the water layer is initially at rest:

$$\eta(x, 0) = \eta_0(x) \equiv 0.$$

The computational domain $[\alpha, \beta]$ is discretized in N equal subintervals, called usually the control volumes. The time step is chosen locally in order to satisfy the following CFL condition [14] used in shallow-water models:

$$\Delta t \leq \frac{\Delta x}{\max\limits_{1 \leq i \leq N} u_i + \sqrt{gd}}.$$

The values of all physical and numerical parameters used in simulations are given in Table 1.

On the left and right boundaries we apply the Neumann-type boundary conditions which do not produce reflections. In any case, we stop the simulation before the generated wave reaches the right boundary. We recall that the gBBM equations (2.10) describes the unidirectional (rightwards, for instance) wave propagation. So, the influence of the left boundary condition is negligible.

Let us describe the constraints that we impose on the shape $\zeta_0(x) \geq 0$ and trajectory $x_0(t)$ of the underwater wavemaker. First of all, we fix the length $2\ell_0$ of this object. Then, we assume that its height is also bounded:

$$\max_{x \in \mathbb{R}} \frac{\zeta_0(x)}{d} \leq a_0.$$

We allow the piston to move during the first T_f s, its motion always starts at the same initial point x_0^0 and it is confined to some wave generation area $[x_0(0), x_f] \subseteq [\alpha, \beta]$:

$$\operatorname{supp} x_0'(t) \subseteq [0, T_f], \qquad x_0(0) = x_0^0, \qquad x_0^0 \le x_0(t) \le x_f, \quad \forall t \in [0, T_f].$$

However, the cost function $J(x_0, \zeta_0)$ is evaluated at time $T > T_f$ so that the generated waveform can evolve further into the desired shape.

Moreover, we require that the piston speed and acceleration are bounded, since too fast motions are difficult to realize in practice because of the gradually increasing energy consumption:

$$\sup_{t \in [0, T_f]} \Big(|x_0'(t)| + \sqrt{\frac{d}{g}} |x_0''(t)| \Big) \le v_f$$

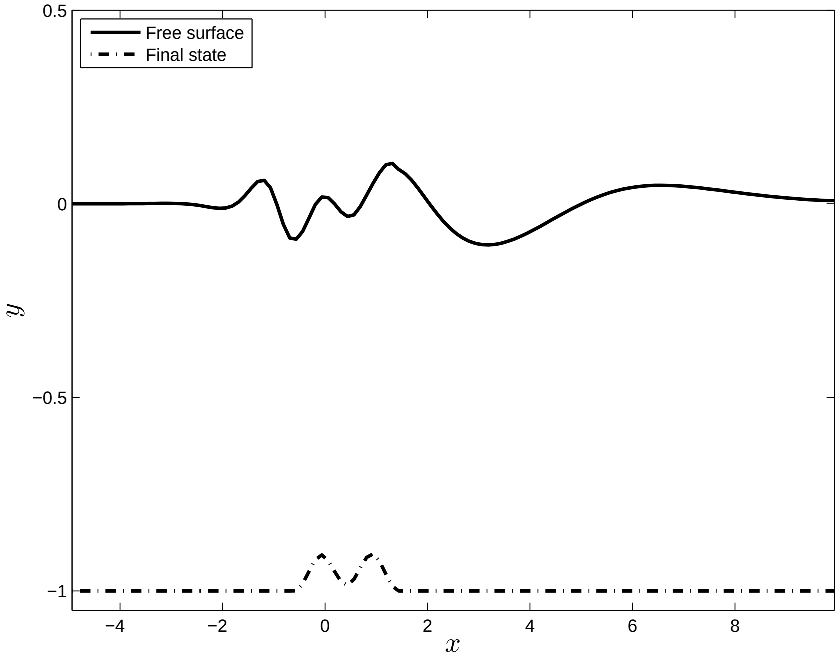
In order to parametrize the wavemaker shape, we use only three degrees of freedom $\zeta_0\big(-\frac{\ell_0}{2}\big)$, $\zeta_0(0)$, $\zeta_0\big(\frac{\ell_0}{2}\big)$ which represent the height of the object in three points equally spaced on the $\operatorname{supp} \zeta_0$. Finally, the continuous shape is reconstructed by applying the interpolation with cubic splines[1] through the following points:

$$(-\ell_0, 0), \quad \Big(-\frac{\ell_0}{2}, \zeta_0\big(-\frac{\ell_0}{2}\big)\Big), \quad (0, \zeta_0(0)), \quad \Big(\frac{\ell_0}{2}, \zeta_0\big(\frac{\ell_0}{2}\big)\Big), \quad (\ell_0, 0).$$

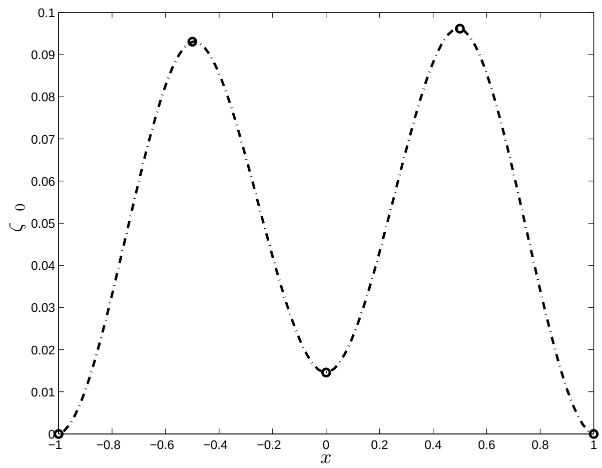
In a similar way we proceed with the parametrization of the piston trajectory $x_0(t)$ which is represented with 4 degrees of freedom (three in the interior of the interval $(0, T_f)$ and the final point $x_0(T_f)$ which is not fixed as in the case of ζ_0). Obviously, more degrees of freedom can be taken into account when it is needed for a speecific application. However, the number of degrees of freedom determines the dimension of the phase space where we find the optimal solution. In examples below we operate in a closed subset of $\mathbb{R}^7$. In order to obtain an approximate solution to our constrained optimization problem, we use the function `fmincon` function of the Matlab Optimization Toolbox. This solver is a gradient-based optimization procedure which uses the SQP algorithm. The iterative process is stopped when the default tolerances are met or the maximal number of iterations is reached. The values of default tolerances are TolCon(tolerance on the constraint violation)= TolFun (tolerance on the function value)=TolX (tolerance on the parameter estimates)=10^{-6} and MaxIter (maximum number of iterations)=200.

As the first numerical example, we minimize the functional $J(x_0, \zeta_0)$ subject to constraints described in above. Basically, this cost-function measures the wave deviation from the still water level in a fixed portion $[a, b]$ of the wave tank. consequently, bigger waves in this interval will provide lower values to the functional J. The result of the numerical optimisation procedure is represented on Figure 3. The free surface elevation computed at the final time T is shown on Figure 3(a). One can see that in the region of interest $[a, b] = [2, 4]$ m we have a big depression wave which is followed by a wave of elevation. We make a conclusion that we succeeded to generate a wave suitable for surfing purposes in artificial environments. The computed shape of the underwater object is shown on Figure 3(b) and its trajectory is represented on Figure 3(c). It is interesting to note that the computed optimal shape is composed of two bumps. The piston trajectory can be conditionally

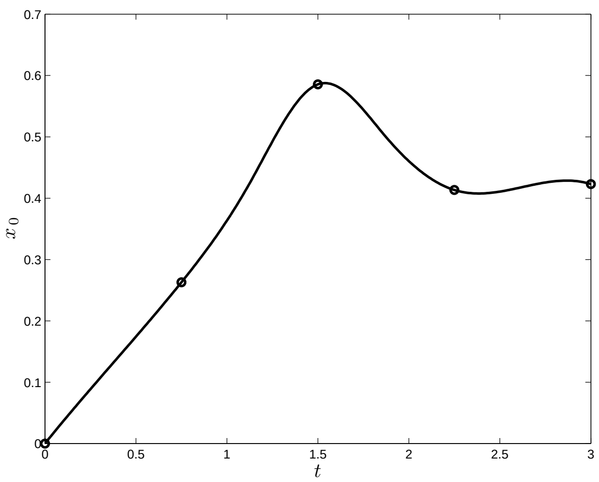
[1]Cubic splines ensure that the interpolant belongs to the class C^2.



(a) Free surface elevation



(b) Piston shape



(c) Piston trajectory

Figure 3. Computed numerically the optimal piston shape and its trajectory which minimize the functional $J(x_0, \zeta_0)$.

decomposed into three parts. During the first 1.25 s we have a stage of slow motion, which is followed by a rapid acceleration and, during the last 0.75 s, we can observe a slight backward motion of the piston before it is frozen in its final point. Then, the wait has $T - T_f = 4$ s to evolve before its *quality* is estimated using the cost function $J(x_0, \zeta_0)$.

Since the choice of the functional to minimize is far from being unique, we decided to perform some additional tests. Instead of maximizing the wave height, one can try to

maximize, for example, the wave steepness in a given portion of the wave tank. In other words, we will minimize the following functional (subject to the same constraints as above):

$$J_1(x_0, \zeta_0) = -\int_{\mathcal{I}} \eta_x(x, T)\,\mathrm{d}x.$$

The result of the numerical optimization procedure is shown on Figure 4. One can see on the free surface snapshot 4(a) that effectively the wave became steeper. The optimal shape of the wavemaker is almost the same as for the functional $J(x_0, \zeta_0)$. However, the piston trajectory is almost monotonic and close to the uniform motion. This solution might be easier to implement in practice.

Finally, we can simply minimize the mismatch between the obtained solution and a fixed desired wave profile:

$$J_2(x_0, \zeta_0) = \int_{\mathcal{I}} \left(\eta(x, T) - \eta_T(x)\right)^2 \mathrm{d}x,$$

where $\eta_T(x)$ is a given function on the interval $\mathcal{I}$. To illustrate this concept, in numerical computations we take the $\mathcal{N}$-wave ansatz put forward by S. Tadepalli & C. Synolakis (1994, 1996) [53, 54]:

$$\eta_T^{(1)}(x) = (x - x_m)\,\mathrm{sech}^2(x - x_m), \qquad \eta_T^{(2)}(x) = -(x - x_m)\,\mathrm{sech}^2(x - x_m).$$

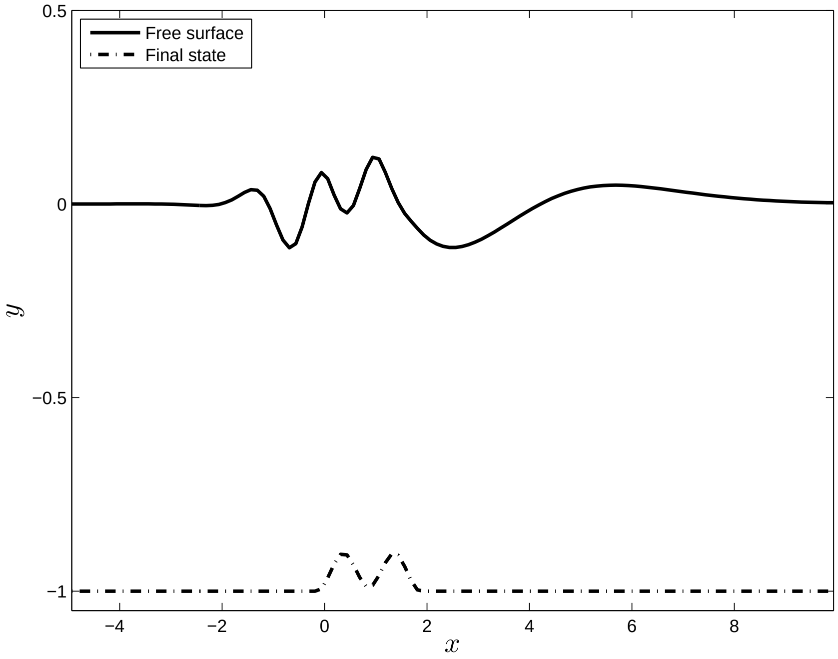
The first profile $\eta_T^{(1)}(x)$ corresponds to the leading elevation $\mathcal{N}$-wave solution (LEN), while the second function $\eta_T^{(2)}(x)$ is a typical leading depression $\mathcal{N}$-wave (LDN). The results of optimization procedures are shown on Figures 5 and 6. One can notice that the resulting optimal shapes of the wavemaker are completely different (see Figures 5(b) and 6(b)). For the surfing applications the LDN wave might be more interesting. It requires also more uniform piston motion comparing to the LEN wave (see Figures 5(c) and 6(c)).

Remark 1. *The arguments used to prove Theorem 2 can be also applied to show the existence of minimizers for the functionals $J_1(x_0, \zeta_0)$ and $J_2(x_0, \zeta_0)$.*

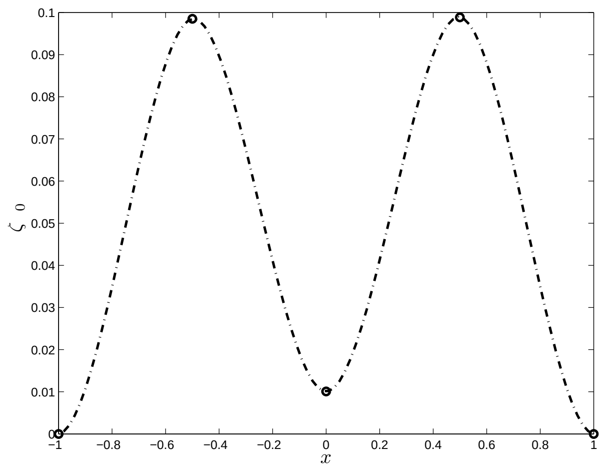
5. Conclusions

In the present work we considered the water wave generation problem by disturbances moving along the bottom. This problem has many important applications going even to the design of artificial surfing facilities [34]. In order to study the formation of water waves due to the motion of the underwater piston, we derived a generalized forced BBM (gBBM) equation. The existence and uniqueness of its solutions were rigorously established. The trajectory of the piston is determined as the solution of a thoroughly formulated optimization problem. The existence of minimizers is also proven. Finally, the theoretical developments of this study are illustrated with numerical examples where we solve several constrained optimization problems with various forms of the cost functional. The resulting solutions are compared and discussed.

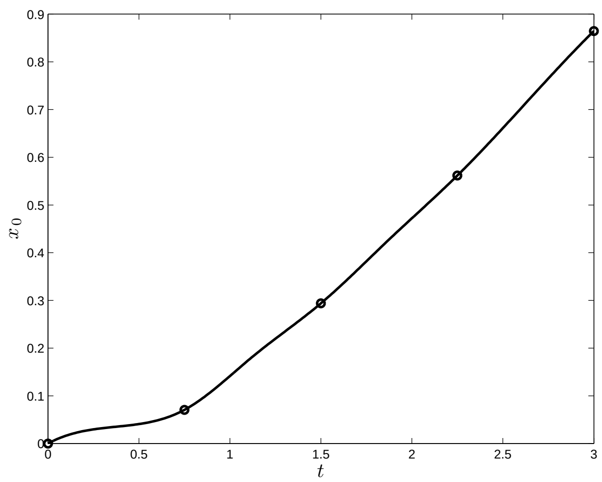
In future studies this problem will be addressed in the context of more complete bidirectional wave propagation models of Boussinesq-type [5, 17, 39, 16]. The optimization



(a) Free surface elevation



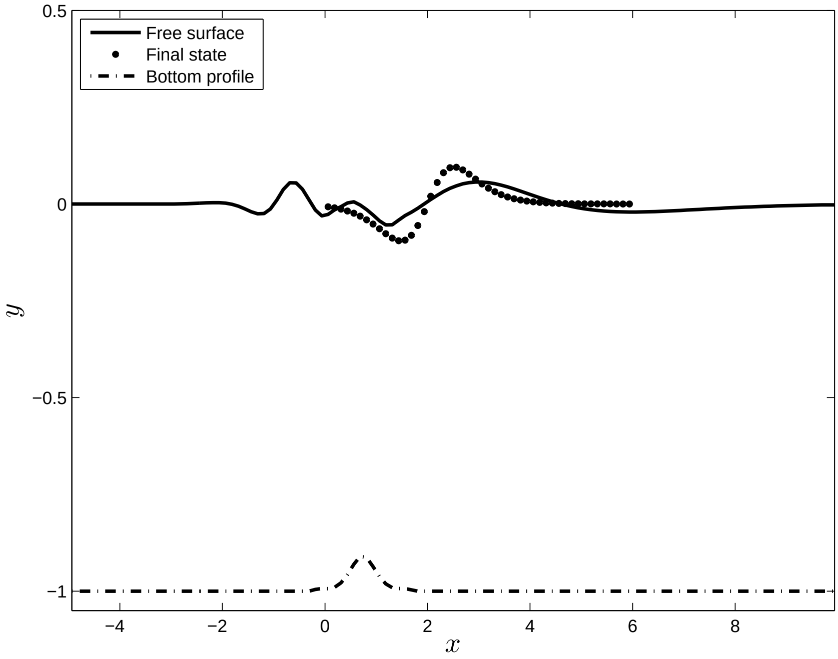
(b) Piston shape



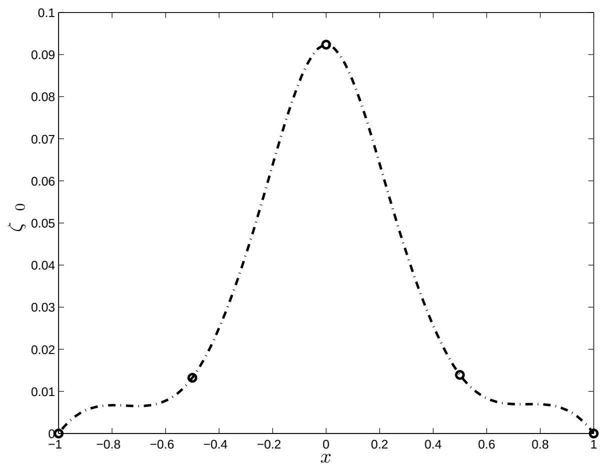
(c) Piston trajectory

FIGURE 4. Computed numerically the optimal piston shape and its trajectory which minimize the functional $J_1(x_0, \zeta_0)$.

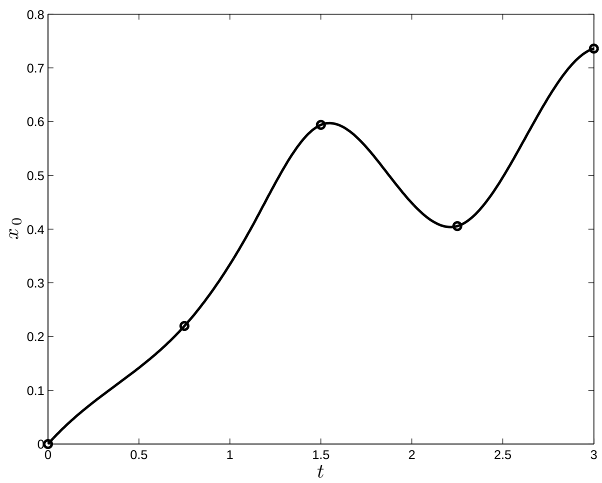
algorithm can be also further improved by evaluating the gradients analytically, for example. From physical point of view, one may want to include some weak dissipative effects for more realistic wave description [18].



(a) Free surface elevation



(b) Piston shape



(c) Piston trajectory

FIGURE 5. Computed numerically the optimal piston shape and its trajectory which minimize the functional $J_2(x_0, \zeta_0)$ and the terminal state $\eta_T^{(1)}(x) = (x - x_m)\,\mathrm{sech}^2(x - x_m)$.

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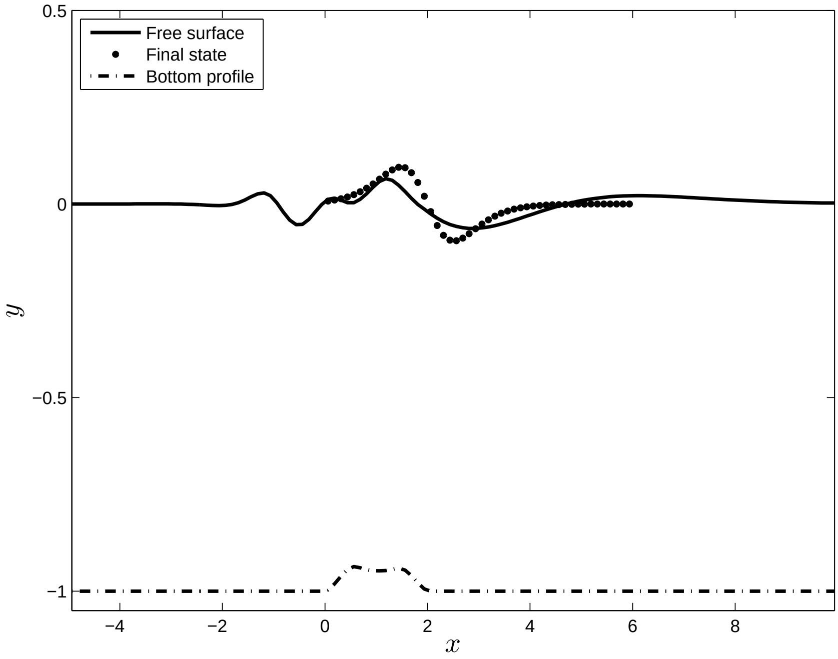
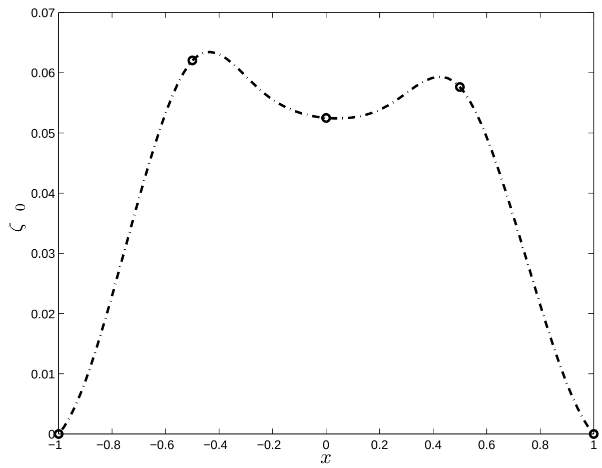
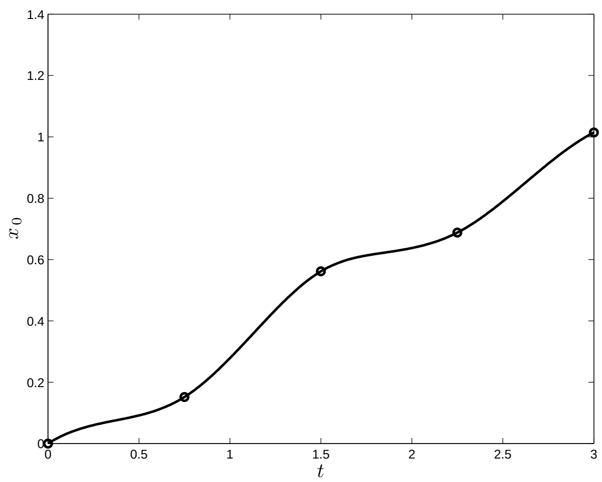


(a) Free surface elevation



(b) Piston shape



(c) Piston trajectory

Figure 6. Computed numerically the optimal piston shape and its trajectory which minimize the functional $J_2(x_0, \zeta_0)$ and the terminal state $\eta_T^{(2)}(x) = -(x - x_m)\,\mathrm{sech}^2(x - x_m)$.

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